An Introduction to the Degree of Commutativity

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1 Measuring Abelianness

Often when trying to learn more about a finite group we check if it is abelian, nilpotent or soluble. These are all, in some sense, measures of abelianness. However we could directly ask the question: what is the probability that two randomnly chosen elements from this group commute? We call this quantity the degree of commutativity.

Definition 1.1. Let G be a finite group. The degree of commutativity of G is

$$d(G) = \frac{1}{|G|^2}|\{(x,y) \in G \times G \mid xy = yx\}|.$$

Any abelian group has degree of commutativity equal to one, however by a well known result of Gustafson, any non-abelian group has $d(G) \leq \frac{5}{8}$.

Theorem 1.2 ([4, p. 1032])**.** *Let G be a non-abelian group then $d(G) \leq \frac{5}{8}$.*

Proof. The degree of commutativity is maximised when the centre is as large as possible. For a non-abelian group $[G : Z(G)] \geq 4$, otherwise if $[G : Z(G)] \leq 3$ then $G/Z(G) \cong C_l = \langle h \rangle$ and so

$$G = \bigcup_{i=0}^{l-1} h^l Z(G) = \langle Z(G), h \rangle$$

is abelian.

For any $g \in G \setminus Z(G)$ then we must have $[G : C_G(g)] \geq 2$. So a group that satisfies $[G : Z(G)] = 4$ and $[G : C_G(g)] = 2$ for all $g \in G \setminus Z(G)$ will have maximal degree of commutativity. Therefore

$$d(G) \leq \left(\frac{1}{4} \cdot 1\right) + \left(\frac{3}{4} \cdot \frac{1}{2}\right) = \frac{5}{8}.$$

$\square$

This upper bound is sharp and achieved by D_8 and Q_8, which notably for later are extraspecial 2-groups.

2 Properties

The degree of commutativity has some very natural properties with respect to subgroups and quotients. Furthermore, it can be expressed as a simple formula in terms of the order of the group and the number of conjugacy classes $k(G)$.

Lemma 2.1 ([7, Lemma 1.1])**.** *The degree of commutativity is related to the number of conjugacy classes of G in the following sense*

$$d(G) = \frac{k(G)}{|G|}.$$

Proof. By the Orbit Stabiliser Theorem, if $x, y \in g^G$ then $|C_G(x)| = |C_G(y)|$. In other words, two elements in the same orbit commute with the same number of elements of G. Let $x_1, \ldots, x_k$ be conjugacy class representatives. Then we can compute a weighted average

$$d(G) = \sum_{i=1}^{k} \frac{|x_i^G|}{|G|} \frac{|C_G(x_i)|}{|G|} = \sum_{i=1}^{k} \frac{1}{|G|} = \frac{k(G)}{|G|}.$$

□

Since the two factors in a direct product compute we can quickly deduce the following.

Lemma 2.2. *Let $G = H \times K$ then $d(G) = d(H)d(K)$.*

A result of Gallagher relates the number of conjugacy classes of a group to the number of conjugacy classes of one of its subgroups. Combining this with the formula for the degree of commutativity, we can relate the degree of commutativity of a group and its subgroups.

Lemma 2.3 ([3, pp. 175–176])**.** *Let $H \leq G$, then $k(H) \geqslant \frac{k(G)}{[G:H]}$ with equality when $G = HC_G(x)$ for some $x \in G$.*

Corollary 2.4. *Let $H \leq G$ then $d(H) \geq d(G)$.*

By comparing the centralisers of elements in a group and its quotient we can deduce the following.

Lemma 2.5 ([3, p. 176])**.** *Let $N \trianglelefteq G$ then $d(G) \leq d(N)d(G/N)$.*

The following lemma is a very useful ingredient in some of the later results.

Lemma 2.6 ([7, Lemma 1.3])**.** *Let G be a non-abelian p-group then $d(G) \leq \frac{p^2+p-1}{p^3}$.*

3 Isoclinisms

In 1940, Hall [5, p. 133] in his paper on ‘The classification of prime-power groups’ introduced an equivalence relation between groups, coined isoclinism. Hall then used this relation to classify groups of order p^n for $n \leq 5$ up to their isoclinism class. Later, in 1995, Lescot [7, Lemma 2.4] connected isoclinism to the degree of commutativity by proving this quantity was preserved between isoclinic groups. And hence, we will shift to studying isoclinic groups.

Before we introduce isoclinisms let $a_G : (G/Z(G))^2 \to G'$ be a map such that $(gZ(G), hZ(G)) \to [g, h]$.

Definition 3.1. Two groups G and H are isoclinic if there exist maps (φ_1, φ_2) such that:

i. $\varphi_1 : G/Z(G) \to H/Z(H)$ is an isomorphism.

ii. $\varphi_2 : G' \to H'$ is an isomorphism.

iii. The following diagram commutes

$$\begin{tikzcd} (G/Z(G))^2 \arrow[r, "\varphi_1 \times \varphi_1"] \arrow[d, "a_G"] & (H/Z(H))^2 \arrow[d, "a_H"] \\ G' \arrow[r, "\varphi_2"] & H'. \end{tikzcd} \tag{1}$$

Since isoclinism is an equivalence relation, we write $G \sim H$ if G and H are isoclinic.

It is challenging to offer the intuition behind why two groups satisfying this definition must have equal degree of commutativity, even after we prove this result in Theorem 3.5. However it is somewhat natural how such a definition could arise. Solubility and nilpotence are important families of groups with 'close to abelian' structure. So for a relation to preserve the degree of commutativity, we will likely want it to preserve the derived length and nilpotency class. Hence requiring isomorphic derived subgroups and central quotients for our relation makes sense. This relation alone is too weak to preserve the degree of commutativity (take for example $G = D_8 \times D_8$ and $H = (C_4 \rtimes Q_8) \rtimes C_2 = \text{SmallGroup}(64, 244)$). Hence we would want to introduce some form of interplay between the central quotients and derived subgroups, and the maps a_G and a_H are a natural choice to aid this.

An immediate consequence of this definition is that isomorphic groups will be isoclinic.

Lemma 3.2 ([7, Lemma 2.1])**.** *If $G \cong H$ then $G \sim H$.*

Another consequence of this definition is that all abelian groups are isoclinic to the trivial group. Isoclinism also preserves the properties we would like, in particular solubility and nilpotence, and in most cases derived length and nilpotency class as well.

Lemma 3.3. *Let G and H be isoclinic groups. If G is soluble of derived length d, then either*

i. G is trivial and, H is abelian and non-trivial,

ii. H is soluble of derived length d.

We omit the proof of the above lemma but prove the result for the nilpotent case.

Lemma 3.4. *Let G and H be isoclinic groups. If G is nilpotent of class n, then either*

i. G is trivial and, H is abelian and non-trivial,

ii. H is nilpotent of class n.

Proof. We first claim, if $G \neq \{e\}$ then G is nilpotent of class n if and only if $G/Z(G)$ is nilpotent of class $n-1$. Suppose $G \neq \{e\}$ is nilpotent of class n and has upper central series given by,

$$\{e\} = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_n = G.$$

Each Z_{i+1} satisfies the relation that $Z_{i+1}/Z_i = Z(G/Z_i)$. Since $Z(G) = Z_1 \subseteq Z_i$ then by the Correspondence Theorem we can consider the subnormal series

$$\{e\} = Z_1/Z(G) \trianglelefteq Z_2/Z(G) \trianglelefteq \cdots \trianglelefteq Z_n/Z(G) = G/Z(G).$$

By the Second Isomorphism Theorem,

$$Z_{i+1}/Z_i \cong \frac{Z_{i+1}/Z(G)}{Z_i/Z(G)} = Z\left(\frac{G/Z(G)}{Z_i/Z(G)}\right) \cong Z(G/Z_i).$$

And hence this is precisely the upper central series for $G/Z(G)$ of length $n-1$. The converse proof is similar.

For the proof of the statement, if we are not in case (i), since $G \sim H$ then $G/Z(G) \cong H/Z(H)$. So $G/Z(H)$ and $H/Z(H)$ are nilpotent of class $n-1$ and so H is nilpotent of class n. $\square$

Most importantly, isoclinism preserves the degree of commutativity.

Theorem 3.5 ([7, Lemma 2.4])**.** *Let G and H be isoclinic groups then $d(G) = d(H)$.*

Proof. We have

$$\begin{aligned}
|G/Z(G))|^2 d(G) &= \frac{1}{|Z(G)|^2} |G|^2 d(G) \\
&= \frac{1}{|Z(G)|^2} |\{(g_1, g_2) \in G \times G \mid g_1 g_2 = g_2 g_1\}| \\
&= \frac{1}{|Z(G)|^2} |\{(g_1, g_2) \in G \times G \mid [g_1, g_2] = e\}| \\
&= \frac{1}{|Z(G)|^2} |\{(g_1, g_2) \in G \times G \mid a_G(g_1 Z(G), g_2 Z(G)) = e\}| \\
&= |\{(x, y) \in G/Z(G) \times G/Z(G) \mid a_G(x, y) = e\}| .
\end{aligned}$$

The final step follows since a_G is constant on cosets of $G/Z(G)$.

$$\begin{aligned}
|G/Z(G))|^2 d(G) &= |\{(x, y) \in G/Z(G) \times G/Z(G) \mid \varphi_2(a_G(x, y)) = e\}| \text{ since } \varphi_2 \text{ is an isomorphism.} \\
&= |\{(x, y) \in G/Z(G) \times G/Z(G) \mid a_H(\varphi_1(x), \varphi_1(y)) = e\}| \text{ since } G \sim H. \\
&= |\{(v, w) \in H/Z(H) \times H/Z(H) \mid a_H(v, w) = e\}| \text{ since } \varphi_1 \text{ is an isomorphism.} \\
&= |H/Z(H))|^2 d(H) \text{ by following the previous steps in reverse.}
\end{aligned}$$

Since $G/Z(G) \cong H/Z(H)$ then it follows $d(G) = d(H)$. □

With some fiddling, we can show the two extraspecial 2-groups of order 8 are isoclinic.

Lemma 3.6. *The groups D_8 and Q_8 are isoclinic.*

Proof. We use the presentations

$$\begin{aligned}
D_8 &= \langle r, s \mid r^4 = s^2 = e, sr = r^{-1}s\rangle, \\
Q_8 &= \langle a, b \mid a^4 = e, a^2 = b^2, ba = a^{-1}b\rangle.
\end{aligned}$$

We have $Z(D_8) = D_8' = \langle r^2\rangle$ and $Z(Q_8) = Q_8' = \langle a^2\rangle$. Furthermore

$$\begin{aligned}
D_8/D_8' &\cong \{\langle r^2\rangle, r\langle r^2\rangle, s\langle r^2\rangle, rs\langle r^2\rangle\}, \\
Q_8/Q_8' &\cong \{\langle a^2\rangle, a\langle a^2\rangle, b\langle a^2\rangle, ab\langle a^2\rangle\}.
\end{aligned}$$

Define the natural maps

$$\begin{aligned}
&\varphi_1 : D_8/Z(D_8) \to Q_8/Z(Q_8) \text{ where } \varphi_1(r^m s^n \langle r^2\rangle) = a^m b^n \langle a^2\rangle, \\
&\varphi_2 : D_8' \to Q_8' \text{ where } \varphi_2(r^2) = a^2.
\end{aligned}$$

These are clearly isomorphisms. These maps commute as in Theorem 3.1 since

$$\varphi_2(a_{D_8}(r^i s^j \langle r^2\rangle, r^k s^l \langle r^2\rangle)) = \varphi_2([r^i s^j, r^k s^l]) = \varphi_2(r^{2i}) = a^{2i}.$$

And

$$a_{Q_8}(\varphi_1(r^i s^j \langle r^2\rangle), \varphi_1(r^k s^l \langle r^2\rangle)) = a_{Q_8}(a^i b^j \langle a^2\rangle, a^k b^l \langle a^2\rangle) = [a^i b^j, a^k b^l] = a^{2i}.$$

Therefore D_8 and Q_8 are isoclinic. □

4 Isoclinisms and Solubility

A surprising yet elegant result is that any group with $d(G) > \frac{1}{12}$ must be soluble. The proof uses the CFSG and some results from representation theory. This result is sharp since $d(A_5) = \frac{1}{12}$.

Lemma 4.1 ([9, pp. 235–251])**.** *Suppose G is a finite, non-abelian, simple group with some irreducible character $\chi \neq \mathbb{1}$ of dimension at most 3. Then G is isomorphic to $A_5, \mathrm{PSL}_2(\mathbb{F}_7)$ or A_6.*

Theorem 4.2 ([8, Theorem 1])**.** *Let G be a group with $d(G) > \frac{1}{12}$, then G is soluble.*

Proof. Suppose G is insoluble, we aim to show that $d(G) \leqslant \frac{1}{12}$. Consider its composition series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}.$$

Then by Theorem 2.5,

$$d(G) \leqslant d(G_1)d(G_0/G_1) \leqslant d(G_2)d(G_1/G_2)d(G_0/G_1) \leqslant \cdots \leqslant \prod_{i=0}^{n-1} d(G_i/G_{i+1}).$$

Since G is insoluble then the composition series must contain a non-abelian factor, say G_j/G_{j+1}. Hence

$$d(G) \leqslant d(G_j/G_{j+1}).$$

Therefore we can reduce to the case G is a non-abelian simple group. We recall that

$$k(G) = \sum_{i \geqslant 1} \rho_i, \text{ and } |G| = \sum_{i \geqslant 1} i^2 \rho_i$$

where ρ_i denotes the number of complex irreducible representations of G of degree i. Since G is a non-abelian, simple group then $G = G'$ and so

$$\rho_1 = [G : G'] = 1.$$

Assume first $\rho_2 = \rho_3 = 0$. Then

$$|G| - 1 = \sum_{i \geqslant 2} i^2 \rho_i = \sum_{i \geqslant 4} i^2 \rho_i \geqslant 16 \sum_{i \geqslant 4} \rho_i = 16(k(G) - 1) = 16(|G|d(G) - 1).$$

Rearanging gives us

$$d(G) \leqslant \frac{1}{16} + \frac{15}{16|G|}.$$

Since G is a simple, non-abelian group then $|G| \geqslant 60$, so in turn

$$d(G) \leqslant \frac{1}{16} + \frac{15}{960} = \frac{5}{64} < \frac{1}{12}.$$

It only remains to prove the theorem in the case there is a non-trivial character of degree at most 3. In this case, by Theorem 4.1, G is isomorphic to either $A_5, PSL_2(\mathbb{F}_7)$ or A_6. These groups have degree of commutativity $\frac{1}{12}, \frac{1}{28}$ and $\frac{7}{360}$ respectively. $\square$

The converse statement is false, the soluble family of groups $\{S_4^n \mid n \in \mathbb{N}\}$ contains groups with arbitrarily small degree of commutativity.

5 Isoclinisms and Nilpotency

In this section we aim to classify all the groups with $d(G) \geq \frac{1}{2}$ up to isoclinism. As a result of this we find the values of the degree of commutativity can achieve can be grouped into one isolated point and a countable family with an accumulation point. A further consequence of this is that any group with $d(G) \geq \frac{1}{2}$ is nilpotent. First, we must add a few tools to our collection.

Lemma 5.1 ([7, Lemma 1.2])**.** *If* $d(G) > \frac{1}{4}$ *then* $|G'| \leq \frac{3}{4d(G)-1}$.

Proof. Again recall that

$$k(G) = \sum_{i\geqslant 1} \rho_i, \text{ and } |G| = \sum_{i\geqslant 1} i^2 \rho_i$$

where ρ_i denotes the number of complex irreducible representations of G of degree i. Then

$$\begin{aligned} |G| - \rho_1 &= \sum_{i\geqslant 2} i^2 \rho_i \\ &\geqslant 4 \sum_{i\geqslant 2} \rho_i \\ &= 4(k(G) - \rho_1) \\ &= 4(|G|d(G) - \rho_1). \end{aligned}$$

There are $[G : G']$ degree 1 characters of G, so by rearranging

$$|G'| \leqslant \frac{3}{4d(G) - 1}.$$

□

Definition 5.2. A group G is called a stem group if $Z(G) \subseteq G'$.

The following result is incredibly useful, but rather arduous to prove.

Theorem 5.3 ([7, Proposition 2.5])**.** *Every group is isoclinic to a stem group.*

Let $c \in (\frac{1}{4}, 1]$. Suppose we want to figure out the isoclinism classes of all G such that $d(G) = c$. By Theorem 5.3 we can assume G is isoclinic to a stem group, i.e. $Z(G) \subseteq G'$. Then by Theorem 5.1 we can bound the order of the derived subgroup, and hence the centre as a function of c. If this bound is small then we can eliminate isomorphism classes of the derived subgroup and centre case by case.

To build up to our result when $d(G) \geq \frac{1}{2}$ we want consider groups that are very close to being a abelian, i.e. small extensions of abelian groups.

Definition 5.4. Let G be a p-group. Then G is an extraspecial p-group if $Z(G)$ is cyclic of order p and $G/Z(G)$ is elementary abelian.

An equivalent characterisation is that G is an extraspecial p-group if and only if G is a p-group with $G' = Z(G) \cong C_p$. By identifying $G/Z(G)$ with the field $\mathbb{F}_p^n$ and $Z(G) = G'$ with $\mathbb{F}_p$, we can then think of the map $a_G : (G/Z(G))^2 \to G'$ as a non-degenerate symplectic form, and hence n is even. Hence all extraspecial p-groups have order p^{1+2m} for some $m \in \mathbb{N}$. For each m, there are two extraspecial p-groups up to isomorphism. They can be differentiated by their exponent (when p is odd, one has exponent p, the other has exponent p^2).

Lemma 5.5. *Suppose* G *is an extraspecial* p*-group of order* p^{1+2m}*. Let* $x \in G\backslash Z(G)$*, then* $|C_G(x)| = p^{2m}$.

Proof. Consider the restricted map $a^*_G(xZ(G),\cdot) : G/Z(G) \to Z(G)$. This is a surjective group homomorphism with

$$\ker a^*_G = \{gZ(G) \in G/Z(G) \mid [x,g] = e\} = C_G(x)/Z(G).$$

Hence by the Homomorphism Theorem and Second Isomorphism Theorem,

$$\frac{G}{C_G(x)} \cong \frac{G/Z(G)}{C_G(x)/Z(G)} \cong Z(G).$$

So $|C_G(x)| = p^{2m}$. □

Using this we can now compute the degree of commutativity of an extraspecial p-group.

Lemma 5.6 ([7, Corollary 3.2])**.** *Suppose G is an extraspecial p-group of order p^{2m+1}, then $d(G) = \frac{p^{2m}+p-1}{p^{2m+1}}$.*

Proof. By Theorem 5.5, for any $x \in G\backslash Z(G)$ then $|C_G(x)| = p^{2m}$. It follows

$$\begin{aligned} p^{2+4m}d(G) &= |G|^2 d(G) \\ &= |G|k(G) \\ &= \sum_{x\in G} |C_G(x)| \text{ by Burnside's Lemma} \\ &= |Z(G)||G| + (|G| - |Z(G)|)p^{2m} \\ &= p \cdot p^{2m+1} + (p^{2m+1} - p)p^{2m} \\ &= p^{2m+1}(p^{2m} + p - 1). \end{aligned}$$

□

We can in fact show that all non-abelian groups of order p^3 are isoclinic. We can do this by first observing that any non-abelian group of order p^3 is an extraspecial p-group. Then we can construct explicit maps to show the groups are isoclinic from the definition, as in the proof of Theorem 3.6.

Theorem 5.7. *Let G and H be two non-abelian groups of order p^3, then $G \sim H$.*

Finally, we can prove the main result of this section, identifying the possible isoclinisms class of groups with $d(G) \geq \frac{1}{2}$.

Theorem 5.8 ([7, Theorem 3.1])**.** *Let G be a group such that $d(G) \geqslant \frac{1}{2}$. Then*

i. $G \sim \{e\}$ is abelian and $d(G) = 1$.

ii. $G/Z(G)$ is elementary abelian of order 2^{2m}, and $d(G) = \frac{1}{2}(1 + 1/4^m) \leqslant \frac{5}{8}$.

iii. $G \sim S_3$ and $d(G) = \frac{1}{2}$.

Proof. Suppose first G is abelian then clearly $d(G) = 1$ and $G \sim \{e\}$. Now suppose G is non-abelian and $d(G) \geqslant \frac{1}{2}$. By Theorem 5.3 we know every group is isoclinic to a stem group, so without loss of generality we can assume G is a stem group, i.e., $Z(G) \subseteq G'$. Then since $d(G) > \frac{1}{4}$ using Lemma 5.1

$$|G'| \leqslant \frac{3}{4d(G) - 1} \leqslant 3.$$

So either $Z(G) = \{e\}$ or $Z(G) = G'$.

Suppose $Z(G) = \{e\}$. Let $n = |G|$ and $m = |E|$ where

$$E = \{g \in G \mid [G : C_G(g)] = 2\}.$$

Note that the number of elements g of G with $[G : C_G(g)] \geqslant 3$ is precisely $n - (m+1)$. It follows

$$\begin{aligned}
\frac{n^2}{2} &\leqslant |G|^2 d(G) \text{ since } d(G) \geqslant \tfrac{1}{2} \\
&= |G| k(G) \\
&= \sum_{x \in G} |C_G(x)| \text{ by Burnside's Lemma} \\
&= n \cdot 1 + \frac{n}{2} \cdot m + \sum_{\substack{x \in G \\ [G:C_G(x)] \geqslant 3}} |C_G(x)| \\
&\leqslant n + \frac{mn}{2} + (n - (m+1)) \frac{n}{3} \\
&= \frac{2n}{3} + \frac{mn}{6} + \frac{n^2}{3}.
\end{aligned}$$

Rearranging gives us

$$n^2 - n(4+m) \leqslant 0.$$

Since n is positive, then we must have $m \geqslant n - 4$. If $n < 10$, then $G \sim S_3$, as this is the only non-abelian group of order less than 10 that has degree of commutativity greater than one half. If $n \geqslant 10$ then

$$m \geqslant n - 4 \geqslant \frac{n}{2} + 1.$$

Let $g \in E$, so then $G/C_G(g) \cong C_2$ is abelian, hence $G' \subseteq C_G(g)$. So for all $h \in G'$ then h commutes with g, in other words $g \in C_G(G')$. Therefore $E \subseteq C_G(G')$. It follows

$$|C_G(G')| \geqslant |E| = m > \frac{n}{2}.$$

This forces $G = C_G(G')$ hence $G' \subseteq Z(G)$. But earlier we assumed $Z(G) = \{e\}$, so $G' = \{e\}$ and G is abelian, a contradiction.

Now suppose $Z(G) = G' \neq \{e\}$, so G is nilpotent. Let G_p denote the unique Sylow p-subgroup of G. By Theorem 2.5 and Theorem 2.6

$$\frac{1}{2} \leqslant d(G) \leqslant d(G_p) d(G/G_p) \leqslant d(G_p) \leqslant \frac{p^2 + p - 1}{p^3}.$$

Hence by simplifying, $p^2 < 2p + 2$ and so

$$(p-1)^2 = p^2 - 2p + 1 < 3.$$

It follows $p = 2$. Since G is nilpotent it must be a 2-group. And by assumption and the earlier bound on the order of the derived subgroup $G' = Z(G) \cong C_2$. Hence G is an extraspecial 2-group and so $G/Z(G)$ is elementary abelian, and its degree of commutativity is given by Theorem 5.6. $\square$

Corollary 5.9. *Let G be a group such that $d(G) > \frac{1}{2}$, then G is nilpotent of class at most* 2.

Proof. By Theorem 5.8, G is isoclinic to $\{e\}$ or an extraspecial 2-group. The result follows by Theorem 3.4 since isoclinism preserves nilpotency class. $\square$

This result is sharp since $d(S_3) = \frac{1}{2}$. The converse also does not hold; the family of nilpotent groups $\{D_8^n \mid n \in \mathbb{N}\}$ contains groups that have an arbitrarily small degree of commutativity.

6 Groups with Large Degree of Commutativity

In the previous section, we proved the strong result that any group G with $d(G) \geqslant \frac{1}{2}$ is either isoclinic to the trivial group, an extraspecial 2-group, or to S_3. Hence the degree of commutativity of a group in this case, can only achieve a value in

$$\left\{1, \frac{1}{2}\right\} \bigcup \left\{\frac{1}{2}\left(1 + \frac{1}{4^m}\right) \mid m \in \mathbb{N}\right\}.$$

Thus it is natural to consider which possible values can the degree of commutativity achieve in the interval $(0, \frac{1}{2})$? Furthermore if $d(G) = \frac{a}{b} \in \mathbb{Q}$, then what can we say about the structure of G and the possible isoclinism classes of G?

In 1979, Rusin [10] classified the possible isoclinism classes of groups with $d(G) \in (\frac{11}{32}, \frac{1}{2})$. And hence the only possible values of the degree of commutativity in this range are

$$\left\{\frac{7}{16}, \frac{11}{27}, \frac{2}{5}, \frac{25}{64}, \frac{3}{8}, \frac{5}{14}\right\}.$$

It is worth noting that the infinite family of extraspecial 3-groups have degree of commutativity in the interval $(\frac{1}{3}, \frac{11}{27}]$. With the smallest order extraspecial 3-group having degree of commutativity $\frac{11}{27}$. We now introduce some tools to offer alternative proofs (from that of Rusin) to determine the isoclinism class of some groups with $d(G) \in (\frac{11}{32}, \frac{1}{2})$. This is based of methods of Buckley and MacHale [2] on groups with $d(G) = \frac{1}{3}$.

Lemma 6.1 ([2, Lemma 13]). *If $Z(G) = \{e\}$, then $|G| \leq |Z(G')||\operatorname{Aut}(G')|$.*

The following result is a direct consequence of nilpotent groups being the direct product of p-groups that each have non-trivial centre.

Lemma 6.2. *Let G be nilpotent, then $p \mid |G|$ if and only if $p \mid |Z(G)|$.*

Lemma 6.3 ([6, Theorem 4.9]). *If $G' \cong C_{2^n}$, then G is nilpotent of class at most $n + 1$.*

Proof. Firstly any cyclic group C_n has a unique subgroup of order m for every $m \mid n$. Therefore if $2 \mid n$ then C_n has a unique involution. Any automorphism of C_n must send the unique involution to itself, hence this involution is central.

We now proceed by induction on n. When $n = 1$, we see that $G' \cong C_2$ and G' contains a unique involution which is central. Therefore $G' \subseteq Z(G)$. It follows $G/Z(G)$ is abelian so G is nilpotent of class 2.

Now assume the induction hypothesis holds, for all $k < n$. So whenever $G' \cong C_{2^k}$ then G is nilpotent of class at most $k + 1$. If $G' \cong C_{2^n}$, since $2 \mid |G'|$ then G' contains a unique involution that is central. So $|G' \cap Z(G)| \geqslant 2$. We have

$$(G/Z(G))' = (G'Z(G))/Z(G) \cong G'/(G' \cap Z(G)).$$

So $(G/Z(G))'$ has order 2^k where $k < n$, since $|G' \cap Z(G)| \geqslant 2$. So $G/Z(G)$ is nilpotent of class at most n by the induction hypothesis. Hence G is nilpotent of class at most $n + 1$. □

Let $H_3(3)$ denote the Heisenberg group over $\mathbb{F}_3$. We already know from Theorem 5.7 that extraspecial 3-groups of order 3^3 are isoclinic. We now prove a stronger statement.

Theorem 6.4. *Let G be a group. Then $d(G) = \frac{11}{27}$ if and only if $G \sim H_3(3)$.*

Proof. Every group is isoclinic to a stem group, so without loss of generality assume $Z(G) \leqslant G'$. Note that since 11 and 27 are coprime, and $d(G) = \frac{k(G)}{|G|}$, so 27 must divide the order of the group. By Theorem 5.1,

$$|G'| \leqslant \frac{3}{4d(G)-1} = \frac{81}{17} < 5.$$

So $|G'| \in \{2, 3, 4\}$. We claim that $|G'| = 3$ so $G \cong C_3$.

Suppose $|G'| \in \{2, 4\}$,

i. If $G' = Z(G)$, then G must be nilpotent. Hence by Theorem 6.2, G must be a 2-group. This is not possible as $27 \mid |G|$.

ii. If $Z(G) = \{e\}$. Then by Theorem 6.1,

$$|G| \leqslant |Z(G')|\,|\mathrm{Aut}(G')| \leqslant 4 \cdot 3 = 12.$$

But $27 \mid |G|$, so we must have $Z(G) \neq \{e\}$.

iii. Suppose $G' \cong C_2 \times C_2$, and $Z(G) = C_2$ then,

$$(G/Z(G))' = G'Z(G)/Z(G) \cong G'/(Z(G) \cap G') \cong G'/Z(G) \cong C_2.$$

Hence, by Theorem 6.3, $G/Z(G)$ is nilpotent, so G is nilpotent also. Therefore by Theorem 6.2, G is a 2-group but this contradicts $27 \mid |G|$.

iv. If $G' \cong C_4$, and $Z(G) = C_2$ a contradiction follows as in iii.

Hence $G' \cong C_3$. And since $Z(G) \leqslant G'$, then either $Z(G) = \{e\}$ or $Z(G) = G'$. We can immediately rule out $Z(G) = \{e\}$ by Theorem 6.1. Since $27 \mid |G|$, then we must have $|G/Z(G)| \geqslant 9$. For contradiction let's assume $|G/Z(G)| \geqslant 10$. Note that $G' = Z(G)$ so G is nilpotent. Hence by Theorem 6.2, G is a 3-group. In particular, note that for any $g \in G \backslash Z(G)$, then $|C_G(g)| \geqslant 3$. It follows

$$d(G) \leqslant \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot \frac{1}{3} = \frac{2}{5} < \frac{11}{27} = d(G).$$

A contradiction, hence $|G/Z(G)| = 9$. So G must have order 27. There are only two non-abelian groups of order 27 and they are both isoclinic by Theorem 5.7. Hence any group with $d(G) = \frac{11}{27}$ is isoclinic to $H_3(3)$. □

Theorem 6.5. *There is no group with* $d(G) \in (\frac{7}{16}, \frac{1}{2})$.

Proof. Suppose G is a group with $d(G) \in \left(\frac{7}{16}, \frac{1}{2}\right)$. Without loss of generality let's assume G is a stem group. Then by Theorem 5.1,

$$|G'| \leqslant \frac{3}{4d(G)-1} < 4.$$

So either $G' \cong C_2$, or $G' \cong C_3$. Theorem 6.1 rules out $Z(G) = \{e\}$, so $Z(G) = G'$ and G is nilpotent. By Theorem 6.2, then G is a 2-group when $Z(G) \cong C_2$, and a 3-group when $Z(G) \cong C_3$. Since G is a stem group then G must be an extraspecial group. By Theorem 5.6, an extraspecial p-group of order p^{2m+1} has degree of commutativity given by

$$d(G) = \frac{p^{2m} + p - 1}{p^{2m+1}} = \frac{1}{p} + \frac{p-1}{p^{2m+1}}.$$

For $p = 2$, then $d(G) > \frac{1}{2}$, so G cannot be a 2-group. And for $p = 3$, then $d(G) \leqslant \frac{11}{27} < \frac{7}{16}$. So G cannot be a 3-group. Hence no group G with $d(G) \in \left(\frac{7}{16}, \frac{1}{2}\right)$ exists. □

For a full classification of the isoclinism classes of groups with $d(G) \in (\frac{11}{32}, \frac{1}{2})$, see [10]. In Section IV, the final part of case two, they miss the isoclinism class of D_{14} which has degree of commutativity $\frac{5}{14}$.

7 Infinite Groups

There is a natural extension of the degree of commutativity to infinite groups, originally introduced by Antolin, Martino and Ventura.

Definition 7.1 ([1, Definition 1.4])**.** Let G be a finitely generated and X a finite generating set. The degree of commutativity of G with respect to X, denoted $dc_X(G)$, is given by

$$dc_X(G) = \limsup_{n\to\infty} \frac{|\{(x,y) \in (\mathbb{B}_X(n))^2\} : xy = yx|}{|\mathbb{B}_X(n)|^2}.$$

Most notably, this definition depends on the generating set X and it is not yet known if it is independent.

Example 7.2. Consider $D_\infty = \langle r, s \mid s^2 = e, rs = sr^{-1}\rangle$, the infinite dihedral group. Set $X = \{r, s, r^{-1}\}$ a generating set for D_∞. Take note for later the integers $\mathbb{Z} \leqslant D_\infty$ are an index 2 abelian subgroup of D_∞. Let us compute $dc_X(D_\infty)$ from the definition. Consider the balls $\mathbb{B}_X(n)$ on the Cayley Graph $\Gamma(D_\infty, X)$. We have

$$\begin{aligned}
\mathbb{B}_X(1) &= \{e, r, r^{-1}, s\},\\
\mathbb{B}_X(2) &= \{e, r, r^{-1}, s, r^2, r^{-2}, rs, r^{-1}s\},\\
&\vdots\\
\mathbb{B}_X(n) &= \{e, r, r^{-1}, s, r^2, r^{-2}, rs, r^{-1}s, \ldots, r^n, r^{-n}, r^{n-1}s, r^{-(n-1)}s\}.
\end{aligned}$$

Hence $|\mathbb{B}_X(n)| = 4n$ for $n \geq 1$.

Next we want to consider how many elements commute inside the ball of radius n. The identity element commutes with all elements in this ball. The element r^i commutes with all possible r^j. And $r^k s$ commutes only with the identity element and itself.

Summarising, we see the identity element commutes with $4n$ elements. There are $2n$ elements of the form r^i and each one commutes with $2n+1$ elements. There are $2n-1$ elements of the form $r^k s$ and these elements commute with two others. It follows

$$\begin{aligned}
dc_X(D_\infty) &= \limsup_{n\to\infty} \left(\frac{4n + 2n(2n+1) + 2(2n-1)}{(4n)^2}\right)\\
&= \limsup_{n\to\infty} \left(-\frac{1}{8n^2} + \frac{5}{8n} + \frac{1}{4}\right)\\
&= \frac{1}{4}.
\end{aligned}$$

In Antolin. Martino and Ventura's paper they prove an analogous result to Gustafson's $\frac{5}{8}$ Theorem.

Theorem 7.3 ([1, Theorem 1.5])**.** *Let G be a finitely generated, residually finite group with subexponential growth. Let X be a generating set for G. Then,*

i. $dc_X(G) > 0$ *if and only if G is virtually abelian.*

ii. $dc_X(G) > \frac{5}{8}$ *if and only if G is abelian.*

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