Normal Coverings of Finite Classical Groups

Jack Gidney

June 2026

UNIVERSITY OF BIRMINGHAM

A Question

Let G be a group and $H < G$ a proper subgroup. Can we have

$$G = \bigcup_{g \in G} gHg^{-1}?$$

A Question

Let G be a group and $H < G$ a proper subgroup. Can we have

$$G = \bigcup_{g \in G} gHg^{-1}?$$

Example

$\mathrm{GL}_n(\mathbb{C}) = \bigcup_{g \in G} gUg^{-1}$ where U is the subgroup of upper triangular matrices.

Jordan's Theorem

Theorem (Jordan, 1872)

No finite group can be covered by conjugates of a proper subgroup, i.e.

$$G \neq \bigcup_{g \in G} gHg^{-1} \text{ for } H < G.$$

Jordan's Theorem

Theorem (Jordan, 1872)

No finite group can be covered by conjugates of a proper subgroup, i.e.

$$G \neq \bigcup_{g \in G} gHg^{-1} \text{ for } H < G.$$

Proof.

- There are $[G : N_G(H)] \leq [G : H]$ conjugates of H in G.

Jordan's Theorem

Theorem (Jordan, 1872)

No finite group can be covered by conjugates of a proper subgroup, i.e.

$$G \neq \bigcup_{g \in G} gHg^{-1} \text{ for } H < G.$$

Proof.

- There are $[G : N_G(H)] \leq [G : H]$ conjugates of H in G.
- So $\left|\bigcup_{g \in G} gHg^{-1}\right| \leq |H||G : H| \leq |G|$.

Jordan's Theorem

Theorem (Jordan, 1872)

No finite group can be covered by conjugates of a proper subgroup, i.e.

$$G \neq \bigcup_{g \in G} gHg^{-1} \text{ for } H < G.$$

Proof.

- There are $[G : N_G(H)] \leq [G : H]$ conjugates of H in G.
- So $\left|\bigcup_{g \in G} gHg^{-1}\right| \leq |H||G : H| \leq |G|$.
- But each conjugate contains the identity, so the size of this union is strictly less than the order of G. □

Normal Coverings

Definition

A set of proper subgroups $\{H_1, \ldots, H_k\}$ of G is a **normal covering** for G if

$$G = \bigcup_{i=1}^{k} \bigcup_{g \in G} gH_i g^{-1}.$$

The **normal covering number** of G is the smallest size of a normal covering of G, denoted $\gamma(\boldsymbol{G})$.

Symmetric and Linear Groups

Example

$\gamma(S_3) = \gamma(S_4) = 2.$

Symmetric and Linear Groups

Example

$\gamma(S_3) = \gamma(S_4) = 2$.

Theorem (Bubboloni-Praeger-Spiga, 2013)

There is $c > 0$ s.t. $cn \leq \gamma(A_n), \gamma(S_n) \leq 2n/3$.

Symmetric and Linear Groups

Example

$\gamma(S_3) = \gamma(S_4) = 2.$

Theorem (Bubboloni-Praeger-Spiga, 2013)

There is $c > 0$ s.t. $cn \leq \gamma(A_n), \gamma(S_n) \leq 2n/3$.

There have been recent improvements on this by Eberhard-Melon (2024) via additive combinatorics.

Symmetric and Linear Groups

Example

$\gamma(S_3) = \gamma(S_4) = 2$.

Theorem (Bubboloni-Praeger-Spiga, 2013)

There is $c > 0$ s.t. $cn \leq \gamma(A_n), \gamma(S_n) \leq 2n/3$.

There have been recent improvements on this by Eberhard-Melon (2024) via additive combinatorics.

Theorem (Britnell-Maróti, 2013)

If $n > 2$ and $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$ then

$$\frac{n}{\pi^2} \leq \gamma(G) = \gamma(G/Z(G)) \leq \frac{n+1}{2}.$$

Proof Ideas for Symmetric vs Linear Groups

Let p be an odd prime.

Example

Let $G = S_p$. Then any element of G is contained in a conjugate of $S_k \times S_{p-k}$ for $1 < k \leq \frac{p-1}{2}$, or in $\mathrm{AGL}_1(p)$.

Proof Ideas for Symmetric vs Linear Groups

Let p be an odd prime.

Example

Let $G = S_p$. Then any element of G is contained in a conjugate of $S_k \times S_{p-k}$ for $1 < k \leq \frac{p-1}{2}$, or in $\mathrm{AGL}_1(p)$.

Example

Let $G = \mathrm{GL}_p(q)$ with q any prime power. Then an element of G stabilises a subspace of dimension $1 \leq k \leq \frac{p-1}{2}$, or is contained in an extension field subgroup.

Finite Simple Groups

Conjecture (Bubboloni-Spiga-Weigel, 2024)

The normal covering number of a finite simple group increases with its 'rank', with one known families of exceptions.

Finite Simple Groups

Conjecture (Bubboloni-Spiga-Weigel, 2024)

The normal covering number of a finite simple group increases with its 'rank', with one known families of exceptions.

Let G be a non-abelian finite simple group.

G	$\gamma(G)$	Reference
$G = A_n$	Grows linearly with n	BPS, 2013
$G = \mathrm{PSL}_n(q)$	Grows linearly with n	BM, 2013
G classical	?	-
G exceptional	?	-
G sporadic	$\gamma(G) \leq 9$	BSW, 2024

The Conjecture

Conjecture (Bubboloni-Spiga-Weigel, 2024)

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\gamma(G) \leq c$ then either

- $|G| \leq f(c)$ *or,*
- G *has Lie rank bounded by* $f(c)$ *or,*
- $G = \mathrm{Sp}_n(q)$ *with q even.*

The Conjecture

Conjecture (Bubboloni-Spiga-Weigel, 2024)

There is a function $f : \mathbb{N} \to \mathbb{N}$ *such that if* $\gamma(G) \leq c$ *then either*

- $|G| \leq f(c)$ *or,*
- G *has Lie rank bounded by* $f(c)$ *or,*
- $G = \mathrm{Sp}_n(q)$ *with* q *even.*

Theorem (Dye, 1979)

Let q *be even. Then*

$$\mathrm{Sp}_n(q) = \bigcup_{g \in G} \mathrm{O}_n^+(q)^g \cup \mathrm{O}_n^-(q)^g.$$

An Extra Family

Theorem

Let $\Omega_n(q) \leq G \leq \mathrm{SO}_n(q)$ *where* n, q *are odd. Then* $\gamma(G) = 3$.

An Extra Family

Theorem

Let $\Omega_n(q) \leq G \leq \mathrm{SO}_n(q)$ *where* n, q *are odd. Then* $\gamma(G) = 3$.

Proof.

1. $A \sim A^{-1}$.
2. Eigenvalues over algebraic closure of $\mathbb{F}_q$ are $\{\lambda_1, \ldots, \lambda_n\}$.
3. $\{\lambda_1, \ldots, \lambda_n\} = \{\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\}$.

An Extra Family

Theorem

Let $\Omega_n(q) \leq G \leq \mathrm{SO}_n(q)$ where n, q are odd. Then $\gamma(G) = 3$.

Proof.

1. $A \sim A^{-1}$.
2. Eigenvalues over algebraic closure of $\mathbb{F}_q$ are $\{\lambda_1, \ldots, \lambda_n\}$.
3. $\{\lambda_1, \ldots, \lambda_n\} = \{\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\}$.
4. Inversion is order two and there are odd number of eigenvalues. Therefore one eigenvalue is ± 1.
5. A stabilises a 1-space $\langle u \rangle$.
6. A is contained in a conjugate of $\mathrm{O}^-_{n-1}(q), \mathrm{O}^+_{n-1}(q)$ or P_1 depending on norm of u.

□

Subgroups of Classical Groups I

Let $r \mid n$ be prime and $G = \mathrm{GL}_n(q)$.

- Extension field type subgroup of prime degree r, denoted E_r. This is a normaliser of $\mathrm{GL}_{n/r}(q^r) \leq \mathrm{GL}_n(q)$.

Subgroups of Classical Groups I

Let $r \mid n$ be prime and $G = \mathrm{GL}_n(q)$.

- Extension field type subgroup of prime degree r, denoted E_r. This is a normaliser of $\mathrm{GL}_{n/r}(q^r) \leq \mathrm{GL}_n(q)$.

Idea:

If V is an n-dimensional vector space over $\mathbb{F}_q$, then V is an n/r-dimensional vector space over $\mathbb{F}_{q^r}$.

Subgroups of Classical Groups I

Let $r \mid n$ be prime and $G = \mathrm{GL}_n(q)$.

- Extension field type subgroup of prime degree r, denoted E_r. This is a normaliser of $\mathrm{GL}_{n/r}(q^r) \leq \mathrm{GL}_n(q)$.

Idea:

If V is an n-dimensional vector space over $\mathbb{F}_q$, then V is an n/r-dimensional vector space over $\mathbb{F}_{q^r}$.

This generalises to the other classical groups.

Subgroups of Classical Groups II

Let G be a classical group.

- Stabiliser type subgroup of a totally isotropic k-dimensional subspace, denoted P_k.
- Stabiliser type subgroup of a non-degenerate k-dimensional subspace (only for symplectic, orthogonal and unitary groups).

Key Lemma

Lemma (Britnell-Maróti, 2013)

Let $r \mid n$ be prime and $A \in \mathrm{GL}_n(q)$. If $A \notin E_r$ then A stabilises a subspace of dimension coprime to r.

Key Lemma

Lemma (Britnell-Maróti, 2013)

Let $r \mid n$ be prime and $A \in \mathrm{GL}_n(q)$. If $A \notin E_r$ then A stabilises a subspace of dimension coprime to r.

Example

The following subgroups form a normal covering for $\mathrm{GL}_6(q)$.

$$\mathcal{N} = \{E_2, P_1, P_3\}.$$

Generalising to the Symplectic Group

Lemma (G, 2026^+)

Let $r \mid n$ be an **odd** *prime and $A \in \mathrm{GSp}_n(q)$. If $A \notin E_r$ then A stabilises a subspace of dimension coprime to r.*

Generalising to the Symplectic Group

Lemma (G, 2026^{+})

Let $r \mid n$ be an **odd** *prime and $A \in \mathrm{GSp}_n(q)$. If $A \notin E_r$ then A stabilises a subspace of dimension coprime to r.*

Example

This no longer holds when $r = 2$. Consider

$$A \sim [C(x^2+1), C(x^6+x^5+x^3+x+1)] \in \mathrm{Sp}_8(3)$$

in rational canonical form. Since A is made up of a block of size two and six, then it can't be contained in E_2. However A only stabilises subspaces of dimension divisible by two.

Symplectic Group Upper Bound I

Theorem (G, 2026^{+})

Let $n = 2^t$ and $\mathrm{Sp}_n(q) \leq G \leq \mathrm{GSp}_n(q)$. Then $\gamma(G) \leq n + 1$.

Symplectic Group Upper Bound I

Theorem (G, 2026^+)

Let $n = 2^t$ and $\mathrm{Sp}_n(q) \leq G \leq \mathrm{GSp}_n(q)$. Then $\gamma(G) \leq n + 1$.

Proof.

All irreducible matrices lie in E_2. The remainder stabilise a totally isotropic or non-degenerate subspace of dimension $k \leq \frac{n}{2}$. □

Symplectic Group Upper Bound II

Theorem (G, 2026^{+})

Let q be odd and suppose that $n = 2^t l$ with $l > 1$ odd. Furthermore, let $\mathrm{Sp}_n(q) \leq G \leq \mathrm{GSp}_n(q)$ and $r_1, \ldots, r_v$ be the distinct primes dividing l. Then

$$\mathcal{N} = \left\{ P_k \mid 1 \leq k \leq \frac{n}{3} \right\} \bigcup \left\{ \mathrm{Sp}_k(q) \times \mathrm{Sp}_{n-k}(q) \mid 1 < k \leq \frac{n}{3} \right\}$$
$$\bigcup \left\{ \mathrm{Sp}_k(q) \times \mathrm{Sp}_{n-k}(q) \mid \frac{n}{3} < k \leq \frac{n}{2},\ (k, l) = 1 \right\} \bigcup \left\{ P_{\frac{n}{2}} \right\}$$
$$\bigcup \{ E_{r_i} \mid 1 \leq i \leq v \}$$

is a normal covering for G.

Current Work: Lower Bounds

Definition

A Singer subgroup of $\mathrm{GL}_n(q)$ is a maximal order irreducible cyclic subgroup in $\mathrm{GL}_n(q)$. A generator of a Singer subgroup is called a Singer cycle.

Current Work: Lower Bounds

Definition

A Singer subgroup of $\mathrm{GL}_n(q)$ is a maximal order irreducible cyclic subgroup in $\mathrm{GL}_n(q)$. A generator of a Singer subgroup is called a Singer cycle.

Singer cycles have order $q^n - 1$, so act regularly on $V \setminus \{0\}$.

Definition

A Singer subgroup of $\mathrm{GL}_n(q)$ is a maximal order irreducible cyclic subgroup in $\mathrm{GL}_n(q)$. A generator of a Singer subgroup is called a Singer cycle.

Singer cycles have order $q^n - 1$, so act regularly on $V \setminus \{0\}$.

One approach to get lower bounds is to find elements that are each contained in a unique maximal subgroup.

Summary

Theorem (Jordan, 1872)

No finite group is a union of conjugates of one proper subgroup.

Theorem (G, 2026^+)

Let q be odd and suppose that $n = 2^t l$ with $l > 1$ odd. Furthermore, let $\mathrm{Sp}_n(q) \leq G \leq \mathrm{GSp}_n(q)$ and $r_1, \ldots, r_v$ be the distinct primes dividing l. Then

$$\mathcal{N} = \left\{P_k \mid 1 \leq k \leq \frac{n}{3}\right\} \bigcup \left\{\mathrm{Sp}_k(q) \times \mathrm{Sp}_{n-k}(q) \mid 1 < k \leq \frac{n}{3}\right\} \bigcup \left\{\mathrm{Sp}_k(q) \times \mathrm{Sp}_{n-k}(q) \mid \frac{n}{3} < k \leq \frac{n}{2},\ (k, l) = 1\right\} \bigcup \left\{P_{\frac{n}{2}}\right\} \bigcup \{E_{r_i} \mid 1 \leq i \leq v\}$$

is a normal covering for G.

Now working towards lower bounds.