

Normal Coverings of Finite Classical Groups

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A Question

Let G be a group and $H < G$ a proper subgroup. Can we have

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Example

$\mathrm{GL}_n(\mathbb{C}) = \bigcup_{g \in G} gUg^{-1}$ where U is the subgroup of upper triangular matrices.

Proof

Theorem

No finite group can be covered by conjugates of a proper subgroup, i.e.

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This result is a consequence of Jordan's Theorem on derangements.

Normal Coverings

Definition

A set of proper subgroups $\{H_1, \dots, H_k\}$ of G is a **normal covering** for G if

$$G = \bigcup_{i=1}^k \bigcup_{g \in G} gH_i g^{-1}$$

The **normal covering number** of G is the smallest number of subgroups that form a normal covering for G , denoted $\gamma(\mathbf{G})$.

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Theorem (Eberhard-Mellon, 2025)

- ▶ $\limsup_{n \text{ even}} \gamma(S_n)/n = 1/4,$
- ▶ $\liminf_{n \text{ even}} \gamma(S_n)/n = 1/6.$
- ▶ *Similar results for n odd and alternating group.*

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Example

$\mathcal{N} = \{GL_1(q^5), P_1, P_2\}$ is a normal covering for $GL_5(q)$.

My Work

Aim: prove this conjecture!

- ▶ Generalize to symplectic, orthogonal and unitary groups.
- ▶ I have similar upper bounds for the symplectic and orthogonal groups.
- ▶ Now working towards lower bounds.